ex1

```
> restart:
> with (student):with(plots):
  with(linalg):
> F:=(x,y)-> x*exp(y)  + cos(x +y) ;
```

$$F := (x, y) \to x\, e^{y} + \cos(x + y) \tag{1}$$

```
> v:=array(1..2,[3,-4]);u := v/norm(v,2); # get that unit vector
```

$$v := \begin{bmatrix} 3 & -4 \end{bmatrix}$$

$$u := \frac{1}{5} v \tag{2}$$

```
> Diff(F(x,y),x)=diff(F(x,y),x);Diff(F(x,y),y)=diff(F(x,y),y);
```

$$\frac{\partial}{\partial x}\left(x\, e^{y} + \cos(x + y)\right) = e^{y} - \sin(x + y)$$

$$\frac{\partial}{\partial y}\left(x\, e^{y} + \cos(x + y)\right) = x\, e^{y} - \sin(x + y) \tag{3}$$

```
> fx:=subs(x=2,y=0,diff(F(x,y),x));
```

$$fx := e^{0} - \sin(2) \tag{4}$$

```
> fy:=subs(x=2,y=0,diff(F(x,y),y));
```

$$fy := 2\, e^{0} - \sin(2) \tag{5}$$

```
> grad1:=array(1..2,[fx,fy]);answer := dotprod(grad1,u); evalf(%);
  # answer to a
```

$$grad1 := \begin{bmatrix} 1 - \sin(2) & 2 - \sin(2) \end{bmatrix}$$

$$answer := -1 + \frac{1}{5} \sin(2)$$

$$-0.8181405146 \tag{6}$$

ex #1b

```
> fx:=subs(x=2,y=0,diff(F(x,y),x));
```

$$fx := e^{0} - \sin(2) \tag{7}$$

```
> fy:=subs(x=2,y=0,diff(F(x,y),y));
```

$$fy := 2\, e^{0} - \sin(2) \tag{8}$$

```
> grad1:=array(1..2,[fx,fy]);
```

$$grad1 := \begin{bmatrix} 1 - \sin(2) & 2 - \sin(2) \end{bmatrix} \tag{9}$$

```
> # 1c the direction of the gradient
```

```
> #1d norm of the gradient
```

```
> norm(grad1,2);
```

$$\sqrt{(1-\sin(2))^2+(2-\sin(2))^2} \tag{10}$$

(11)

```
> #1e a vector perpendicular to grad1:=array(1..2,[fx,fy]);
> perp1:=array(1..2,[fy,-fx]);
```

$$perp1 := \left[\begin{array}{cc} 2-\sin(2) & -1+\sin(2) \end{array}\right] \tag{12}$$

```
> dotprod(grad1,perp1);evalf(%);
```

$$(1-\sin(2))\,(2-\sin(2)) + (2-\sin(2))\,(-1+\sin(2))$$

$$0. \tag{13}$$

ex2

```
>  restart:
> with (student):with(plots):
  with(linalg):
> F:=(x,y,z)-> x^2 + y^2 + z -9  ;
```

$$F := (x,y,z) \to x^2+y^2+z-9 \tag{14}$$

```
> # tangent plane at P=(1,2,4)
```

```
> Diff(F(x,y,z),x)=diff(F(x,y,z),x);Diff(F(x,y,z),y)=diff(F(x,y,
  z),y);Diff(F(x,y,z),Z)=diff(F(x,y,z),z);
```

$$\frac{\partial}{\partial x}\left(x^2+y^2+z-9\right) = 2x$$

$$\frac{\partial}{\partial y}\left(x^2+y^2+z-9\right) = 2y$$

$$\frac{\partial}{\partial Z}\left(x^2+y^2+z-9\right) = 1 \tag{15}$$

```
> Fx:=subs(x=1,y=2,z=4,diff(F(x,y,z),x));Fy:=subs(x=1,y=2,z=4,diff
  (F(x,y,z),y));Fz:=subs(x=1,y=2,z=4,diff(F(x,y,z),z));
```

$$Fx := 2$$

$$Fy := 4$$

$$Fz := 1 \tag{16}$$

```
> v:=array(1..3,[Fx,Fy,Fz]);Po:=array(1..3,[x-1,y-2,z-4]);
```

$$v := \left[\begin{array}{ccc} 2 & 4 & 1 \end{array}\right]$$

$$Po := \left[\begin{array}{ccc} x-1 & y-2 & z-4 \end{array}\right] \quad (17)$$

```
> dotprod(v,Po)=0;# the tangent plane
```

$$-14 + 2\,\overline{x} + 4\,\overline{y} + \overline{z} = 0 \quad (18)$$

#3

```
>  restart:
> with (student):with(plots):
> f:=(x,y)-> -x^2 +x*y -2*y -y^2 + 4 ;
```

$$f := (x, y) \to -x^2 + y\,x - 2\,y - y^2 + 4 \quad (19)$$

```
> xf:=diff(f(x,y),x);yf:=diff(f(x,y),y); solve({xf=0,yf=0},{x,y});
```

$$xf := -2\,x + y$$

$$yf := x - 2 - 2\,y$$

$$\left\{x = -\frac{2}{3}, y = -\frac{4}{3}\right\} \quad (20)$$

```
> disc:= (x,y)-> diff(f(x,y),x,x)*diff(f(x,y),y,y)-diff(f(x,y),x,
  y)*diff(f(x,y),x,y);DXY:=eval(fxx*fyy-fxy*fxy);;
```

$$disc := (x, y) \to \left(\frac{\partial^2}{\partial x^2} f(x, y)\right)\left(\frac{\partial^2}{\partial y^2} f(x, y)\right) - \left(\frac{\partial^2}{\partial y\,\partial x} f(x, y)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x, y)\right)$$

$$DXY := fxx\,fyy - fxy^2 \quad (21)$$

```
> subs(x=-2/3,y= -4/3,disc(x,y));
```

$$3$$

$$3 \quad (22)$$

```
> subs(x=-2/3,y= -4/3,diff(f(x,y),x,x)); # says a local max
```

$$-2 \quad (23)$$

$$3 \quad (24)$$

$$-2$$

ex 4a

```
> restart:with (student):W:= (x,y,z)-> x*y+z;
```

$$W := (x, y, z) \to y\,x + z \quad (25)$$

```
> X:= (t)-> cos(t);Y:= (t)-> sin(t);Z:= (t)-> (t);
```

$$X := t \to \cos(t)$$

$$Y := t \to \sin(t)$$

$$Z := t \to t \quad (26)$$

```
> Diff(W(x,y,z),x)*Diff(X(t),t)+Diff(W(x,y,z),y)*Diff(Y(t),t) +
  Diff(W(x,y,z),z)*Diff(Z(t),t); diff(W(x,y,z),x)*diff(X(t),t)+
  diff(W(x,y,z),y)*diff(Y(t),t) + diff(W(x,y,z),z)*diff(Z(t),t);
```

$$\left(\frac{\partial}{\partial x}(y\,x + z)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\cos(t)\right) + \left(\frac{\partial}{\partial y}(y\,x + z)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\sin(t)\right) + \left(\frac{\partial}{\partial z}(y\,x + z)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\,t\right)$$

$$-y\sin(t)+x\cos(t)+1 \tag{27}$$

```
> subs(x=cos(t),y=sin(t),z=t,%);simplify(%);
```

$$3 \tag{28}$$

$$-\sin(t)^2+\cos(t)^2+1$$
$$2\cos(t)^2 \tag{29}$$

```
> W1:=t -> 2*cos(t^2);diff(W1(t),t); #ex 4b
```

$$W1:=t\to 2\cos(t^2)$$
$$-4\sin(t^2)\,t \tag{30}$$

```
> # ex5 not using chain rule
```

$$3 \tag{31}$$

```
> restart:
> with (student):with(plots):
  :
> f:=(x,y)-> x^2 + y^2    ;
> X:= (u,v)-> u + v;Y:= (u,v)-> u - v;
```

$$f:=(x,y)\to x^2+y^2$$
$$X:=(u,v)\to u+v$$
$$Y:=(u,v)\to u-v \tag{32}$$

$$3 \tag{33}$$

```
> Diff(f(x,y),x,x) + Diff(f(x,y),y,y)=diff(f(x,y),x,x)+diff(f(x,
  y),y,y);
```

$$\frac{\partial^2}{\partial x^2}\left(x^2+y^2\right)+\frac{\partial^2}{\partial y^2}\left(x^2+y^2\right)=4 \tag{34}$$

$$\tag{35}$$

```
> Diff(f(x,y),x)*Diff(X(u,v),u)+Diff(f(x,y),y)*Diff(Y(u,v),v);
  diff(f(x,y),x)*diff(X(u,v),u)+diff(f(x,y),y)*diff(Y(u,v),u);
```

$$\left(\frac{\partial}{\partial x}\left(x^2+y^2\right)\right)\left(\frac{\partial}{\partial u}\left(u+v\right)\right)+\left(\frac{\partial}{\partial y}\left(x^2+y^2\right)\right)\left(\frac{\partial}{\partial v}\left(u-v\right)\right)$$
$$2x+2y \tag{36}$$

```
> subs(x=u+v,y=u-v,%);simplify(%);
```

$$4u$$
$$4u \tag{37}$$

```
> W1:=u ->4*u;diff(W1(u),u); #ex 5b not using chain rule
```

$$W1:=u\to 4u$$
$$4 \tag{38}$$

ex6

```
> restart:with (student):with(linalg):
> f:=(x,y,z)-> exp(3*x+4*y)*cos(5*z);
```

(39)

$$f := (x, y, z) \to e^{3x+4y}\cos(5z) \tag{39}$$

```
> Diff(f(x,y,z),x,x) + Diff(f(x,y,z),y,y) + Diff(f(x,y,z),z,z)=
  diff(f(x,y,z),x,x)+diff(f(x,y,z),y,y)+diff(f(x,y,z),z,z);
```

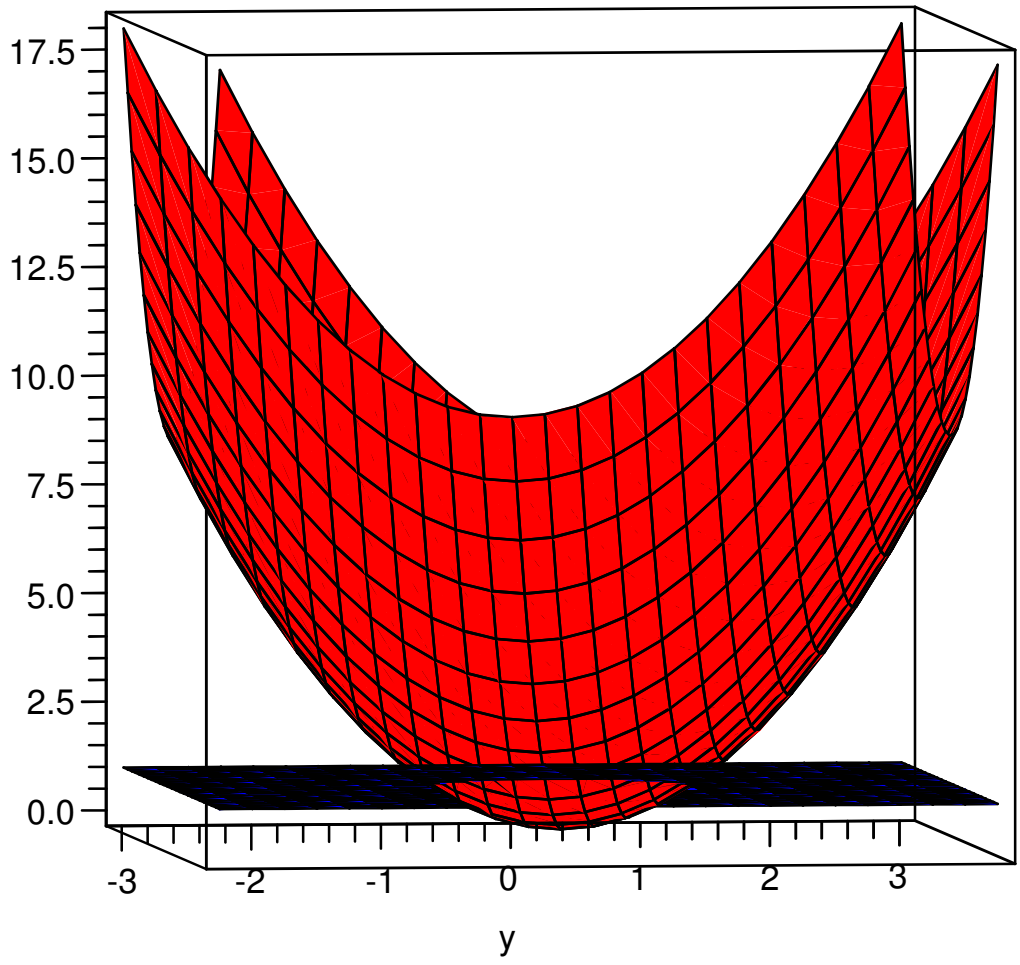
$$\frac{\partial^2}{\partial x^2}\left(e^{3x+4y}\cos(5z)\right) + \frac{\partial^2}{\partial y^2}\left(e^{3x+4y}\cos(5z)\right) + \frac{\partial^2}{\partial z^2}\left(e^{3x+4y}\cos(5z)\right) = 0 \tag{40}$$

ex7 it is #2

```
> # ex 8
> restart:with (student):with(linalg):with(plots):
> F:=(x,y)-> x^2+y^2;plot1 :=plot3d(x^2 + y^2,x=-3..3,y=-3..3,
  color = red):
```

$$F := (x, y) \to x^2 + y^2 \tag{41}$$

```
>  plot2:=plot3d(1,x=-3..3,y=-3..3,color = blue):
> display(plot1,plot2,axes=boxed);
```




```
> r: t -> (t,sqrt(1-t^2),1);
```

$$t \to \left(t, \sqrt{1-t^2}, 1\right) \tag{42}$$

```
> v: t -> (1, -t/sqrt(1-t^2),0);
> subs(t=a,%);
```

$$t \to \left(1, -\frac{t}{\sqrt{1-t^2}}, 0\right)$$

$$a \to \left(1, -\frac{a}{\sqrt{1-a^2}}, 0\right) \tag{43}$$

```
> # ex 9
> restart:with(student):with(plots):
> f:=(x,y)-> -x^4 +4*x*y  -y^4 + 1 ;
```

$$f := (x, y) \to -x^4 + 4\,y\,x - y^4 + 1 \tag{44}$$

```
> xf:=diff(f(x,y),x);yf:=diff(f(x,y),y); solve({xf=0,yf=0},{x,y});
```

$$xf := -4\,x^3 + 4\,y$$

$$yf := 4\,x - 4\,y^3$$

$$\{x=0, y=0\}, \{x=1, y=1\}, \{x=-1, y=-1\}, \{x=-RootOf(\_Z^2+1, label=\_L4), y = RootOf(\_Z^2+1, label=\_L4)\}, \{x=RootOf(-RootOf(\_Z^2+1, label=\_L3)+\_Z^2, label=\_L5)\ RootOf(\_Z^2+1, label=\_L3), y=RootOf(-RootOf(\_Z^2+1, label=\_L3)+\_Z^2, label=\_L5)\} \tag{45}$$

```
> # UGH ! it says 3 real roots
```

$$x2 := -1$$

$$y2 := -1 \tag{46}$$

```
> disc:= (x,y)-> diff(f(x,y),x,x)*diff(f(x,y),y,y)-diff(f(x,y),x,
  y)*diff(f(x,y),x,y);DXY:=eval(fxx*fyy-fxy*fxy);;
```

$$disc := (x, y) \to \left(\frac{\partial^2}{\partial x^2} f(x, y)\right)\left(\frac{\partial^2}{\partial y^2} f(x, y)\right) - \left(\frac{\partial^2}{\partial y\,\partial x} f(x, y)\right)\left(\frac{\partial^2}{\partial y\,\partial x} f(x, y)\right)$$

$$DXY := fxx\,fyy - fxy^2 \tag{47}$$

```
> subs(x=0,y= 0,disc(x,y)); # saddle point
```

$$-16 \tag{48}$$

```
> subs(x=1,y= 1,disc(x,y));subs(x=1,y= 1,diff(f(x,y),x,x)); # says
  a local max
```

$$128$$

$$-12 \tag{49}$$

```
> subs(x=-1,y= -1,disc(x,y));subs(x=1,y= 1,diff(f(x,y),x,x)); #
  says a local max
```

$$128$$

$$-12 \tag{50}$$

```
> #EX 10
> restart:
> with (student):with(plots):
  with(linalg):
> F:=(x,y,z)-> x^2 /9+ y^2 + z^2/9 -3  ;
```

$$F := (x, y, z) \to \frac{1}{9} x^2 + y^2 + \frac{1}{9} z^2 - 3 \tag{51}$$

```
> # tangent plane at P=(-3,1,-3)

> Diff(F(x,y,z),x)=diff(F(x,y,z),x);Diff(F(x,y,z),y)=diff(F(x,y,
  z),y);Diff(F(x,y,z),Z)=diff(F(x,y,z),z);
```

$$\frac{\partial}{\partial x}\left(\frac{1}{9} x^2 + y^2 + \frac{1}{9} z^2 - 3\right) = \frac{2}{9} x$$

$$\frac{\partial}{\partial y}\left(\frac{1}{9} x^2 + y^2 + \frac{1}{9} z^2 - 3\right) = 2 y$$

$$\frac{\partial}{\partial Z}\left(\frac{1}{9} x^2 + y^2 + \frac{1}{9} z^2 - 3\right) = \frac{2}{9} z \tag{52}$$

```
> Fx:=subs(x=-3,y=1,z=-3,diff(F(x,y,z),x));Fy:=subs(x=-3,y=1,z=-3,
  diff(F(x,y,z),y));Fz:=subs(x=-3,y=1,z=-3,diff(F(x,y,z),z));
```

$$Fx := -\frac{2}{3}$$

$$Fy := 2$$

$$Fz := -\frac{2}{3} \tag{53}$$

```
> v:=array(1..3,[Fx,Fy,Fz]);Po:=array(1..3,[x+3,y-1,z +3]);
```

$$v := \begin{bmatrix} -\frac{2}{3} & 2 & -\frac{2}{3} \end{bmatrix}$$

$$Po := \begin{bmatrix} x+3 & y-1 & z+3 \end{bmatrix} \tag{54}$$

```
> dotprod(v,Po)=0;# the tangent plane
```

$$-6 - \frac{2}{3}\overline{x} + 2\overline{y} - \frac{2}{3}\overline{z} = 0 \tag{55}$$

```
> # ex11

> restart:with (student):w:= (x,y,z)-> x^4*y + y^2*z^3;
```

$$w := (x, y, z) \to x^4 y + y^2 z^3 \tag{56}$$

```
> X:= (r,s,t)-> r*s*exp(t);Y:= (r,s,t)-> r*s^2*exp(-t);Z:= (r,s,t)
  -> r*s*sin(t);
```

$$X := (r, s, t) \to r s e^t$$

$$Y := (r, s, t) \to r s^2 e^{-t}$$

$$Z := (r, s, t) \to r\, s \sin(t) \tag{57}$$

```
> Diff(w(x,y,z),x)*Diff(X(r,s,t),s)+Diff(w(x,y,z),y)*Diff(Y(r,s,
  t),s) +Diff(w(x,y,z),z)*Diff(Z(r,s,t),s);
```

$$\left(\frac{\partial}{\partial x}\left(x^4 y + y^2 z^3\right)\right)\left(\frac{\partial}{\partial s}\left(r\, s\, e^t\right)\right) + \left(\frac{\partial}{\partial y}\left(x^4 y + y^2 z^3\right)\right)\left(\frac{\partial}{\partial s}\left(r\, s^2 e^{-t}\right)\right) + \left(\frac{\partial}{\partial z}\left(x^4 y + y^2 z^3\right)\right)\left(\frac{\partial}{\partial s}\left(r\, s \sin(t)\right)\right) \tag{58}$$

```
> diff(w(x,y,z),x)*diff(X(r,s,t),s)+diff(w(x,y,z),y)*diff(Y(r,s,
  t),s) +diff(w(x,y,z),z)*diff(Z(r,s,t),s);
```

$$4 x^3 y\, r\, e^t + 2\left(x^4 + 2 y z^3\right) r\, s\, e^{-t} + 3 y^2 z^2 r \sin(t) \tag{59}$$

```
>
```

```
> subs(x=r*s*exp(t),y=r*s*exp(-t),z=r*s*sin(t),%);simplify(%);
```

$$4 r^5 s^4 \left(e^t\right)^4 e^{-t} + 2\left(r^4 s^4 \left(e^t\right)^4 + 2 r^4 s^4 e^{-t} \sin(t)^3\right) r\, s\, e^{-t} + 3 r^5 s^4 \left(e^{-t}\right)^2 \sin(t)^3$$

$$r^5 s^4 e^{-t}\left(4 e^{4t} + 2 s e^{4t} + 4 s e^{-t} \sin(t)^3 + 3 e^{-t} \sin(t)^3\right) \tag{60}$$

ex12

```
> restart:with(student):with(linalg):with(VectorCalculus):
> T:=(x,y,z)-> 80/(1 +x^2+2*y^2+3*z^2);
```

$$T := (x, y, z) \to 80\, \frac{1}{1 + x^2 + 2 y^2 + 3 z^2} \tag{61}$$

```
>  grad(T(x,y,z), vector([x,y,z])); # direction of gradient
```

$$\left[-\frac{160\, x}{\left(1 + x^2 + 2 y^2 + 3 z^2\right)^2} \quad -\frac{320\, y}{\left(1 + x^2 + 2 y^2 + 3 z^2\right)^2} \quad -\frac{480\, z}{\left(1 + x^2 + 2 y^2 + 3 z^2\right)^2}\right] \tag{62}$$

```
> norm(%,2);
```

$$160 \left(\left|\frac{x}{\left(1 + x^2 + 2 y^2 + 3 z^2\right)^2}\right|^2 + 4\left|\frac{y}{\left(1 + x^2 + 2 y^2 + 3 z^2\right)^2}\right|^2 + 9\left|\frac{z}{\left(1 + x^2 + 2 y^2 + 3 z^2\right)^2}\right|^2\right)^{1/2} \tag{63}$$

```
> #ex13 dis in calss
> # ex 14 did in class
```

```
>
```